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Digital Signal Processor:

Power Measurements

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Power measurement is fundamental to transmission quality testing. This measurement need is extended to signals represented by digital bit streams. Accurate and precise measurements over a 60-dB range have been made using the digital signal processor. One algorithm that has been used measures the power of fixed-length sample sequences. A second algorithm yields periodically updated power measurements of infinitely long sample sequences, but with slightly increased measurement ripple and frequency restriction. Theoretical expectations for measurement variation in the fixed-length measurements of noise power are also discussed.

I. INTRODUCTION

The measurement of power is fundamental to transmission quality testing. Power measurements of single tones, such as the milliwatt standard, are used to adjust transmission levels. Multiple-tone power measurements are used in nonlinear distortion testing. Examples of power measurements of band-limited noise are return-loss and *C*-message weighted noise measurements.

This paper gives an overview, discusses the theoretical accuracy and precision of digital noise power measurements, and presents some results using the digital signal processor (DSP) A3990 for making power measurements.

II. OVERVIEW

The measurement of power will be presented following the block diagram of Fig. 1. Since the incoming signal is generally encoded for bit compression, the signal samples first must be decoded to linear

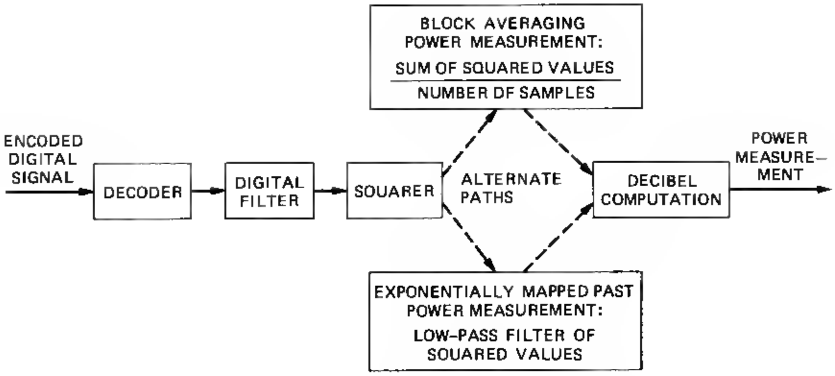

Fig. 1—Time domain technique of power measurement.

samples, as indicated in the first block of the figure. For μ-255 encoding and decoding, the DSP has a dedicated instruction set and associated circuitry.

The decoded digital signal is scaled and passed through a digital filter. This paper discusses measurements with flat weighting, with C-message weighting, and through a C-notched filter.

Note that Fig. 1 depicts essentially a time-domain approach, which should be contrasted with the frequency-domain approach depicted in Fig. 2. These two approaches are tied together by Parseval's Theorem:

$$\text{Signal Power} = \lim_{\mathscr{T}\to\infty} \frac{1}{\mathscr{T}} \int_0^{\mathscr{T}} y^2(t)\, dt = \int_0^{\infty} S(w)\, dw = R(0), \tag{1}$$

where

$\mathscr{T}$ = averaging interval,
$y(t)$ = analog signal,
$S(w)$ = the power-density spectrum, and
$R(0)$ = autocorrelation function at zero lag, i.e., the dc component of $y^2(t)$.

The first integral in eq. (1) is implemented as in Fig. 1; the second, as in Fig. 2. Implementation of the latter is not discussed in this paper.

To compute the first integral, the sample values must be squared, as indicated in Fig. 1. Two different methods of determining the power from the squared signal were used.

The first method is the straightforward approach. If the incoming

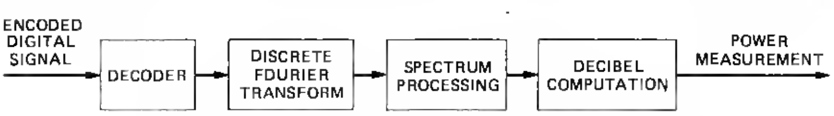

Fig. 2—Frequency domain technique of power measurement.

signal is considered stationary, the power can be approximated from the first integral of eq. (1) as follows:

$$R(0) \cong \frac{1}{\mathscr{T}} \int_0^{\mathscr{T}} y^2(t)\, dt \cong \frac{1}{NT} \sum_{i=0}^{N-1} y_i^2 T = \frac{1}{N} \sum_{i=0}^{N-1} y_i^2, \tag{2}$$

where

N = the number of samples,

y_i = ith signal sample, and

T = interval between the samples ($T = 125\ \mu s$ in most voice telephone applications).

This approach is termed block averaging (BA) in this paper.

To use BA, N must be chosen large enough so that the measurement variation is within the required tolerance. In the next section, the probability of measuring noise power within certain confidence levels for different values of N is derived.

The second method for extracting the dc component employs a convolution of the squared sample values with the infinite impulse response of a first-order, low-pass filter. This algorithm, described in Appendix B, has been termed exponentially mapped past (EMP).

The EMP is not as applicable as BA because of the extra frequency components generated by squaring a signal. Because of the sampled nature of the signals, some components can be aliased into the pass-band of the EMP low-pass filter and, thus, yield measurement ripples. For example, the EMP parameters discussed in this paper yield a ripple of ±0.4 dB for a 15-Hz tone and a ripple of less than ±0.1 dB for an 80-Hz tone.

Once the dc component has been extracted, it can be converted to a dB measurement before it is reported. A method for calculating the required logarithms with the DSP is described in Appendix C.

Currently, the BA program reports a dB computation once per block, i.e., once every N samples, where N = 4096. However, when EMP is used, linear-to-dB conversions can be made more frequently. After a conversion is made in the current EMP program, the next conversion can be made after another three samples have been read from the input buffer. The rate with which conversions can be made and reported during EMP power measurements depends on several variables, as explained in Appendix D.

For display or printout routines, a binary number representing a dB level can be converted to binary-coded decimal (BCD). A BCD routine was used to yield the BA signal-to-quantizing noise ratio measurements described below. This routine is not discussed in this paper.

III. THEORETICAL PRECISION OF NOISE POWER MEASUREMENTS

In this section the noise power measurement precision that is theoretically possible by digital power measurements is presented.[1]

Consider the finite set of noise samples $n_0, n_1, \ldots n_{N-1}$. The ac power of these N samples is expressed as

$$P_n(N) = \frac{1}{N} \sum_{i=0}^{N-1} (n_i - \bar{n})^2, \tag{3}$$

where

$$\bar{n} = \frac{1}{N} \sum_{j=0}^{N-1} n_j$$

is the dc component of the noise.

How large should N be in order that the estimated power $P_n(N)$ be within plus or minus some δ_n (in dB) of the actual power? That is, for a given N, what is the probability $P\{\ \}$ that

$$\left| 10 \log \frac{P_n(N)}{P_{\text{ref}}} - \lim_{M\to\infty} 10 \log \frac{P_n(M)}{P_{\text{ref}}} \right| < \delta_n, \tag{4}$$

where P_{ref} is any reference power? The following analysis assumes that the noise is Gaussian, but results are in most cases applicable to other types of noise (e.g., quantizing or coding noise).

Assume the noise sample n_i to be an independent, zero-mean, Gaussian random variable with finite variance σ^2. Then $P_n(N)$ is an estimate of σ^2, with expected value $= \dfrac{N-1}{N}\sigma^2$ (see Ref. 2). To the accuracy required for the BA program, $P_n(N)$ is effectively an unbiased estimate for $N > 100$. For a given δ_n, eq. (4) becomes

$$\left| 10 \log \frac{P_n(N)}{P_{\text{ref}}} - 10 \log \frac{\sigma^2}{P_{\text{ref}}} \right| < \delta_n \tag{5}$$

or

$$\sigma^2 10^{-\delta_n/10} < P_n(N) < \sigma^2 10^{\delta_n/10}. \tag{6}$$

Now if the probability density function $f_n(\alpha)$ of $P_n(N)$ can be found, then the probability $P\{\ \}$ of satisfying the inequality (5) is

$$P\left\{ \left| 10 \log \frac{P_n(N)}{P_{\text{ref}}} - 10 \log \frac{\sigma^2}{P_{\text{ref}}} \right| < \delta_n \right\} = \int_{\sigma^2 10^{-\delta_n/10}}^{\sigma^2 10^{\delta_n/10}} f_n(\alpha)\, d\alpha. \tag{7}$$

In Appendix A the density function is derived along with the probability. The result is

$$P\{|\mathcal{N}_m - \mathcal{N}_a| < \delta_n\} \cong \int_{t_1}^{t_2} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt, \tag{8}$$

where

$$\mathcal{N}_m = 10 \log \frac{P_n(N)}{P_{\text{ref}}}, \qquad \mathcal{N}_a = 10 \log \frac{\sigma^2}{P_{\text{ref}}}, \qquad N > 30,$$

$$l_2 = \sqrt{2(10)^{\delta_n/10}N} - \sqrt{2N-3},$$

and

$$l_1 = \sqrt{2(10)^{-\delta_n/10}N} - \sqrt{2N-3}.$$

For a graphical representation of eq. (8), see Fig. 3. For example, with 500 samples the probability that the noise power measurement precision is within ±0.2 dB is 53 percent. To meet standard specifications for noise and signal power measurements,[3] N was chosen to be 4096, yielding a precision of ±0.5 dB.

IV. TEST RESULTS

The ability to measure power precisely with both the BA and EMP schemes can be seen by comparing BA and EMP measurements with the actual signal-to-quantizing noise ratios (SNRs) of ideally-encoded sine waves at levels ranging from 3 to −64 dBm. Such encoded sine waves have inherent quantizing noise frequency components across the voiceband range of frequencies. To make SNR measurements, the C-notched and C-message filters are generally used.

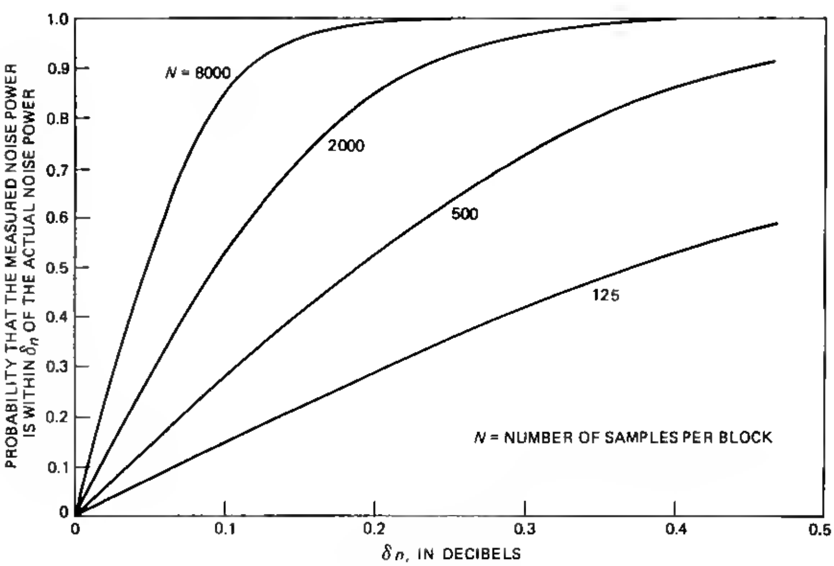

Fig. 3—Theoretical noise power measurement precision.

The C-notched filter is actually two cascaded filters, each implemented with three cascaded, second-order sections. The first filter, the C-message filter, has approximately unity gain from 1000 to 2500 Hz, and its attenuation increases gradually on either side of the passband to 54.7 dB at 60 Hz and to 13.7 dB at 3900 Hz. The second filter, the notch filter, is designed for attenuation of more than 50 dB from 989 to 1020 Hz. The frequency response of the digital C-notched filter was measured by both the EMP measurement routine and by an analog power meter. The results, limited by quantizing noise, appear in Fig. 4.

To determine the SNR of an encoded signal between 1004 and 1020 Hz, the C-message weighted measurement must be subtracted from the C-notched measurement. By means of a real-time developmental tool, the DSPMATE, SNR measurements were made with the BA and EMP programs using ideal, encoded sine waves at 1015.625 Hz. The BA and EMP measurements yielded a range of ±0.5 dB, which was within the theoretical noise power measurement precision. In Fig. 5, the BA derived SNRS are plotted against the actual SNRS.

In order to retain significant bits at low power levels, the programs were modified when the test signal powers were below −27 dBm. After μ-to-linear conversion, sample values from signals above −27 dBm in power were divided by 4, while sample values from signals below −27 dBm in power were multiplied by 8. The DSP can be programmed to choose the appropriate scaling.

Because of quantizing noise, the maximum SNR is approximately 40 dB. Thus, the maximum SNR in Fig. 5 is comparable to the quantizing noise floor of Fig. 4.

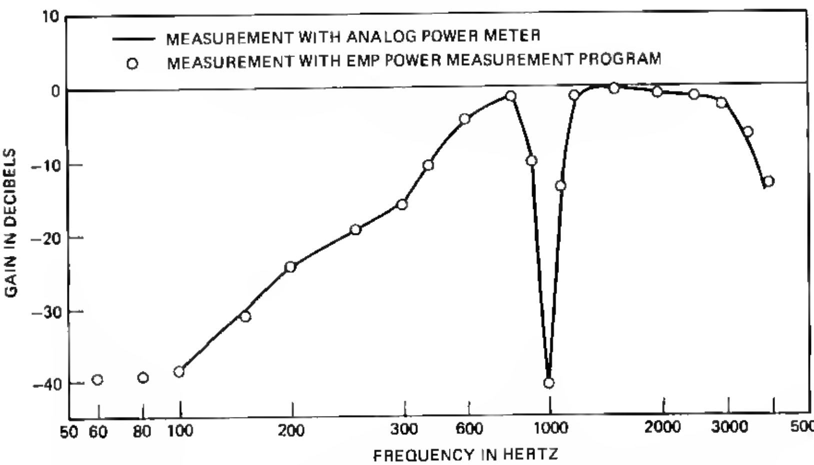

Fig. 4—Frequency response of a digital C-notched filter. (Measured signals are μ-255 coded.)

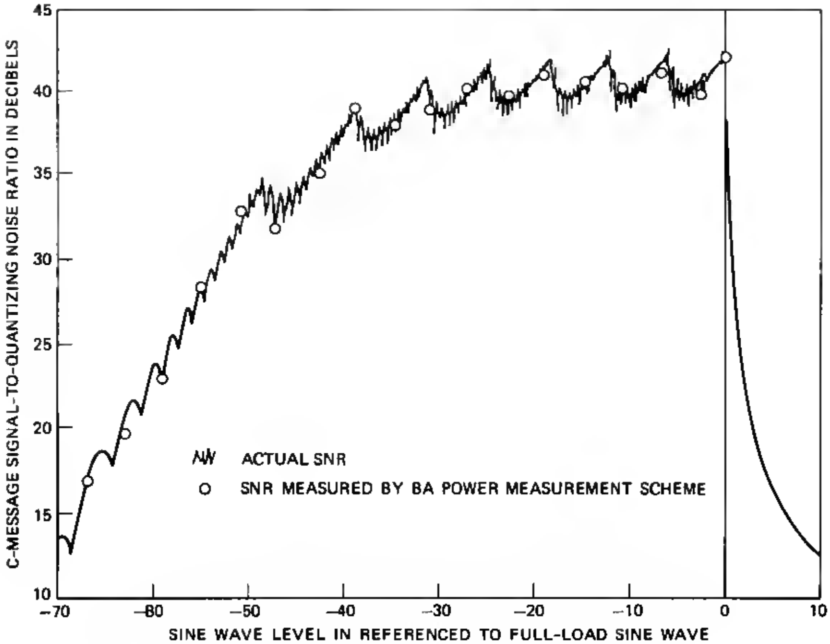

Fig. 5—Measured versus actual SNRs for μ-255 encoding. (Zero dB full-load sine wave = 3.17 dBm.)

V. CONCLUSION

Two reliable methods for measuring power with the DSP have been presented: the BA approach, and the EMP approach. Both of the approaches, when examined mathematically, have no significant bias in their expected values. The accuracy, therefore, is as good as the digital samples representing the signals being measured.

The precision of signal power, noise power, and SNR measurements were investigated. Digital signal processing measurements of these parameters showed that:

(*i*) In general, signal power measurements were precise to within 0.1 dB.

(*ii*) With appropriate scaling, SNR measurements were precise to within 0.5 dB over a 60-dB range.

(*iii*) Block-averaging noise power measurements all fell within theoretical limits.

In the EMP program, frequencies in the range 80 to 3920 Hz yielded measurement ripples less than ±0.1 dB, and frequencies in the range 30 to 3970 Hz yielded ripples less than ±0.2 dB. These measurements were made with an EMP convolution having a 3-dB cutoff at 2.489 Hz. However, these ripples were about twice as large as the measurement ripples from the BA program.

Two advantages of EMP over BA are compactness of code and ability to update a measurement 1365 times as often. Frequent updating may aid in identifying particular types of problems and, hence, aid in problem sectionalization.

APPENDIX A

Derivation of Noise Power Measurement Precision

In the following analysis, the probability density function of $P_n(N)$ is presented and used to find, in computable form, the probability that the ac noise power $P_n(N)$ is within some δ_n (in dB) of the noise variance σ^2.

Recall the definition of ac noise power:

$$P_n(N) = \frac{1}{N} \sum_{i=0}^{N-1} (n_i - \bar{n})^2.$$

As a result of assumptions that n_i is a zero-mean, Gaussian random variable with finite variance σ^2, the probability density function of $P_n(N)$ (see Ref. 4) is

$$f_n(\alpha) = \frac{\alpha^{(N-3)/2} e^{-\alpha N/2\sigma^2}}{2^{(N-1)/2}(\sigma/\sqrt{N})^{N-1}\Gamma[(N-1)/2]}, \tag{9}$$

which is a chi-square density. Hence, the probability $P\{\ \}$ that $P_n(N)$ is within δ_n of the actual noise power is

$$P\left\{\left|10\log\frac{P_n(N)}{P_{\text{ref}}} - 10\log\frac{\sigma^2}{P_{\text{ref}}}\right| < \delta_n\right\} = \int_{\sigma^2 10^{-\delta_n/10}}^{\sigma^2 10^{\delta_n/10}} f_n(\alpha)\, d\alpha. \tag{10}$$

Using the substitution $t = \alpha N/\sigma^2$ yields

$$P\left\{\left|10\log\frac{P_n(N)}{P_{\text{ref}}} - 10\log\frac{\sigma^2}{P_{\text{ref}}}\right| < \delta_n\right\}$$

$$= \int_{10^{-\delta_n/10}N}^{10^{\delta_n/10}N} \frac{t^{(N-3)/2} e^{-t/2}\, dt}{2^{(N-1)/2}\Gamma[(N-1)/2]}. \tag{11}$$

For $N > 30$, this probability can be expressed in terms of the normal probability integral:[5]

$$P\left\{\left|10\log\frac{P_n(N)}{P_{\text{ref}}} - 10\log\frac{\sigma^2}{P_{\text{ref}}}\right| < \delta_n\right\} \cong \int_{l_1}^{l_2} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt, \tag{12}$$

where

$$l_2 = \sqrt{2(10)^{\delta_n/10}N} - \sqrt{2N-3}$$

and

$$l_1 = \sqrt{2(10)^{-\delta_n/10}N} - \sqrt{2N-3}.$$

Eq. (8) and Fig. 3 in the text follow.

APPENDIX B

Derivation of EMP

The result of passing an analog signal $y^2(t)$ through an analog filter with impulse response $g(t)$ is $P(t)$, where

$$P(t) = \int_{-\infty}^{\infty} y^2(u)g(t-u)\,du. \tag{13}$$

Let T be the sampling period of the digital signal $y^2(nT)$, or $y^2(n)$ for short. The result of passing $y^2(n)$ through a digital filter with impulse response $h(n)$, is $P(n)$, where

$$P(n) = \sum_{k=-\infty}^{\infty} y^2(k)h(n-k). \tag{14}$$

An analog, first-order, low-pass filter uses

$$g(t) = \begin{cases} A\,e^{-t/\tau}, & t \geq 0 \\ 0, & t < 0, \end{cases} \tag{15}$$

where τ is the time constant of the filter. Such a filter has a 3-dB cutoff at $(2\pi\tau)^{-1}$ Hz.

If the impulse invariance $h(n) = g(nT)$ is used to form an equivalent digital filter, the corresponding impulse response is

$$h(n) = \begin{cases} A\,e^{-nT/\tau}, & n \geq 0 \\ 0, & n < 0, \end{cases} \tag{16}$$

where T is the sampling period. Because of sampling, aliasing is introduced, but the effects of the aliased components are negligible if the cutoff frequency is low.

For ease in notation, let $m = T/\tau$. From eqs. (14) and (16),

$$P(n) = \sum_{k=-\infty}^{\infty} y^2(k)h(n-k),$$

or

$$P(n) = \sum_{k=-\infty}^{n} y^2(k)A\,e^{-m(n-k)}. \tag{17}$$

From this, a simple recursive relationship for $P(n)$ can be developed:

$$P(n+1) = \sum_{k=-\infty}^{n+1} y^2(k)A\, e^{-m(n+1-k)}$$
$$= e^{-m} \sum_{k=-\infty}^{n+1} y^2(k)A\, e^{-m(n-k)}$$
$$= e^{-m}\, P(n) + e^{-m} y^2(n+1)A\, e^{m} \tag{18}$$

or

$$P(n+1) = e^{-m}\, P(n) + Ay^2(n+1). \tag{19}$$

In eq. (19), A should be chosen to ensure unity gain at dc. To determine A, let

$$y^2(n) = \begin{cases} L, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{and} \quad P(-1) = 0. \tag{20}$$

Then,

$$P(n+1) = e^{-m}P(n) + AL, \tag{21}$$

which implies that $P(n)$ is a geometric series:

$$P(n) = AL \sum_{k=0}^{n} e^{-mk}$$
$$= AL \frac{1 - e^{-m(n+1)}}{1 - e^{-m}}, \tag{22}$$

which approaches L for n approaching infinity if $A = 1 - e^{-m}$.

The EMP power measurement $P(n)$ can, thus, be obtained by the recursion formula

$$P(n) = e^{-m}P(n-1) + (1 - e^{-m})y^2(n), \tag{23}$$

where $m = T/\tau$.

If the range of the signal power to be measured is large and no automatic gain control is to be used, then some double-precision arithmetic has to be done to save the least-significant bits resulting from sums and products. In particular, the result of squaring the input samples nearly doubles the number of significant bits in the accumulator. Therefore, all bits resulting from squaring should be saved. In addition, EMP, which follows the squaring, should be implemented with double precision.

This implementation is facilitated by representing e^{-m}, which is nearly unity for a low cutoff frequency, by $1 - 2^{-R}$, where R is an integer. The time constant τ is

$$\tau = -T/\ln(1 - 2^{-R}). \tag{24}$$

For $R = 9$, $\tau = 63.9$ ms; and the 3-dB cutoff is 2.489 Hz, which yields

power measurements with ripples of less than ±0.1 dB at frequencies between 80 and 3920 Hz.

To implement EMP in double precision, the stored value $P(n - 1)$ must occupy two storage locations. Conveniently, one location could contain its integral part, while the other, its fractional part.

APPENDIX C

Using the DSP to Compute the Natural Logarithm of a Number

Using the DSP, the logarithm of up to a 40-bit number P can be computed to an accuracy of 0.001. Suppose that

$$\begin{aligned} \ln P &= \ln[\mathcal{M}(2^{\mathcal{E}})] \\ &= \ln \mathcal{M} + \mathcal{E} \ln 2, \end{aligned} \tag{25}$$

where $\mathcal{M}$ is a real number and $\mathcal{E}$ is an integer. Then, $\ln P$ can be computed from

(*i*) a series expansion on $\mathcal{M}$ plus

(*ii*) a table of multiples of $\mathcal{E} \ln 2$.

If $0 < \mathcal{M} \le 2$, the following expansion can be used:

$$\ln \mathcal{M} = (\mathcal{M} - 1) - \frac{(\mathcal{M} - 1)^2}{2} + \frac{(\mathcal{M} - 1)^3}{3} - \frac{(\mathcal{M} - 1)^4}{4} + \cdots \tag{26}$$

As indicated in Fig. 6, if $0.68 \le \mathcal{M} \le 1.36$, then $\ln \mathcal{M}$ can be computed to an accuracy of ±0.001 with only four terms. Since the upper bound on $\mathcal{M}$ is twice its lower bound, $\mathcal{E}$ can be determined by repeated scalings of P by 2.

APPENDIX D

Rate of Linear-to-dB Conversions in the EMP Program

In the current EMP program, a linear-to-dB conversion of the power measurement is made after every third sample. Although every sample is used to update the linear power measurement, there is insufficient time to make a conversion to dB after every update.

The number of samples, S, that must be used to update the linear power measurement S times and to make one linear-to-dB conversion can be determined from the following relation:

$$S(R + U) + C < ST, \tag{27}$$

where

R = the time to check whether the input buffer is full and, if so, to read it,

U = the time to update the linear power measurement using the new sample,

C = the time to make a linear-to-dB conversion of the power measurement, and

T = the sampling interval.

Then S can be any integer greater than $C/T - R - U$. In Fig. 7, the case for $S = 3$ is depicted. In this figure, L is the time necessary to load the input buffer.

As shown in Fig. 7, the second sample in each group of three must be read before the third sample begins to be loaded. Expressed symbolically, this means

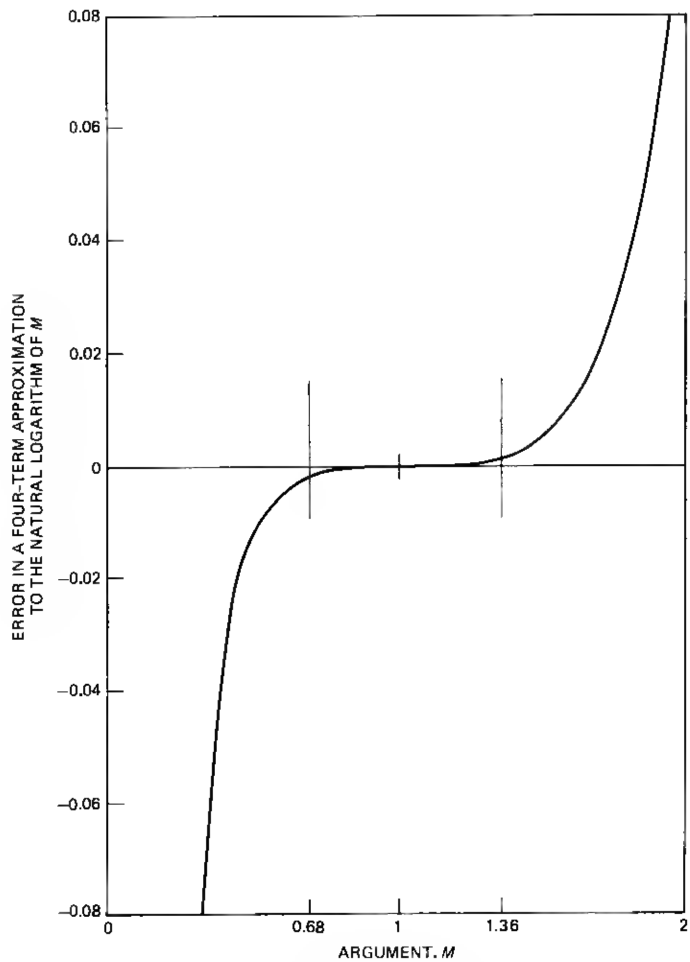

Fig. 6—Performance of a four-term polynomial approximation to the natural logarithm.

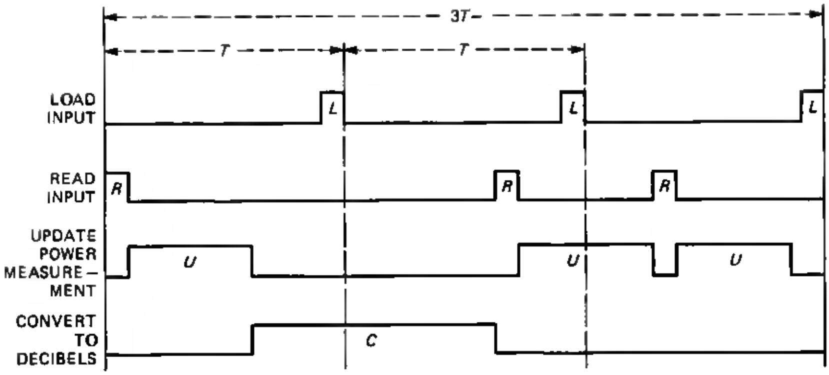

Fig. 7—Timing diagram showing the rate of linear-to-dB conversions in the EMP program.

$$R + U + C + R < T + T - L,$$

or

$$C < 2T - L - 2R - U. \tag{28}$$

If eq. (28) were not satisfied by C, then the second sample in each group of three would have to be stored during the linear-to-dB conversion and then the sample value returned after the conversion was complete.

In the current EMP program, the upper bound on C was sufficiently high that, with $S = 3$, each linear-to-dB conversion could be followed by a conversion to BCD.

V. ACKNOWLEDGMENT

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